Differential Calculus for Dirichlet forms: The measure-valued gradient preserved by image *

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Abstract

In order to develop a differential calculus for error propagation (cf [3]) we study local Dirichlet forms on probability spaces with carré du champ Γ – i.e. error structures – and we are looking for an object related to Γ which is linear and with a good behaviour by images. For this we introduce a new notion called the measure valued gradient which is a randomized square root of Γ. The exposition begins with inspecting some natural notions candidate to solve the problem before proposing the measure-valued gradient and proving its satisfactory properties.

1 Preamble

Our main purpose being to study images, in order to avoid unessential difficulties, we restrict us to Dirichlet forms defined on probability spaces. On a probability space $(W,\ \mathcal{W},\ m)$ let us consider a local Dirichlet form $(\mathbb{D}, \mathcal{E})$

*This work was presented at the meeting on Stochastic Analysis and Potential Theory, St Priest de Gimel, 1-6 sept 2002 with carré du champ operator Γ. This is equivalent (cf.[4], [6]) to the data of

(1) a dense sub-vector space $\mathbb{D}$ of $L^2(W,\ \mathcal{W},\ m)$,

(2) a symmetric positive bilinear operator Γ from $\mathbb{D}\times\mathbb{D}$ into $L^1(m)$ satisfying the following functional calculus : if $u\in\mathbb{D}^m,\ v\in\mathbb{D}^n,$ and F,G are Lipschitz and C^1 from $\mathbb{R}^m$ [resp. $\mathbb{R}^n$] into $\mathbb{R}$ then $F(u)\in\mathbb{D}$ and $G(v)\in\mathbb{D}$ and

$$\Gamma[F(u),G(v)]=\sum_{i,j}F_i'(u)G_j'(v)\Gamma[u_i,v_j]\qquad m\text{-a.e.}$$

(3) and such that the form $\mathcal{E}$ given by

$$\mathcal{E}[u,v]=\frac{1}{2}\int\Gamma[u,v]\,dm$$

is *closed* in $L^2(W,\mathcal{W},m)$, i.e. $\mathbb{D}$ is complete with the norm

$$\|\ .\ \|_{\mathbb{D}}=\Big(\|\ .\ \|^2_{L^2(m)}+\mathcal{E}[\ .\ ]\Big)^{1/2}.$$

We write always $\Gamma[u]$ for $\Gamma[u,u]$. A term $S=(W,\ \mathcal{W},\ m,\ \mathbb{D},\ \Gamma)$ satisfying properties (1) (2) (3) is called an error structure. The notion of error structure is stable by products, even infinite products, and this feature gives easily error structures on spaces of stochastic processes like the Wiener space (cf. books [4], [6] and [3], and examples of application [1], [2]).

Such a term $(W,\ \mathcal{W},\ m,\ \mathbb{D},\ \Gamma)$ is also easily transported by images : If $X\in\mathbb{D}^d$, let us consider the space $\mathcal{C}^1\cap Lip(\mathbb{R}^d,\mathbb{R})$ of functions u of class $\mathcal{C}^1$ and Lipschitz from $\mathbb{R}^d$ into $\mathbb{R}$ (which are such that $u\circ X\in\mathbb{D}$), then the term $S_X=(\mathbb{R}^d,\ \mathcal{B}(\mathbb{R}^d),\ X_*m,\ \mathbb{D}_X,\ \Gamma_X)$ where X_*m is the image measure of m by X, $\mathbb{D}_X$ is the closure of $\mathcal{C}^1\cap Lip(\mathbb{R}^d,\mathbb{R})$ for the norm $\|u\|_{\mathbb{D}_X}=\|u\circ X\|_{\mathbb{D}}$ and

$$\Gamma_X[u](x)=\mathbb{E}[\Gamma[u\circ X]\mid X=x]$$

satisfies still properties (1) (2) (3), i.e. is still an error structure.

The question we attempt to answer here, is to find an object related to Γ which be linear and preserved by image. Let us first look at some existing objects in the literature.

The generator $(A,\ \mathcal{D}A)$ of the strongly continuous semigroup on $L^2(W,\ \mathcal{W},\ m)$ associated with the error structure is a linear operator and is transformed by image into the generator of the image structure by a relation similar to that defining the image of Γ :

$$\forall f \ : \ f \circ X \in \mathcal{D}A \qquad A_X[f](x) = \mathbb{E}[A[f \circ X] \mid X = x]$$

(cf. [4] chapter V prop. 1.1.7 and 1.1.8) but the calculations with A involve non linear operations because of the presence of Γ: if $f \in \mathcal{C}^2_{bb}$ and if $\Gamma[X_i, X_j] \in L^2(m)$ we have

$$A[f \circ X] = \sum_i A[X_i] \frac{\partial f}{\partial x_i}(X) + \frac{1}{2} \sum_{i,j} \Gamma[X_i, X_j] \frac{\partial^2 f}{\partial x_i \partial x_j}(X).$$

A Dirichlet-gradient or shortly D-gradient for the error structure $(W,\ \mathcal{W},\ m,\ \mathbb{D},\ \Gamma)$ is defined with an auxiliary separable Hilbert space H as a linear map D from $\mathbb{D}$ into $L^2(m, H)$ s.t.

(i) $\quad \forall U \in \mathbb{D} \qquad \|D[U]\|^2_H = \Gamma[U].$

Such an operator satisfies necessarily

(ii) $\quad \forall F \in Lip,\ \forall U \in \mathbb{D} \quad D[F \circ U] = F' \circ U.D[U]$

(iii) $\quad \forall F \in \mathcal{C}^1 \cap Lip(\mathbb{R}^d),\ \forall U \in \mathbb{D}^d \quad D[F \circ U] = \sum_i F'_i \circ U.D[U_i].$

Any error structure admits a D-gradient as soon as $\mathbb{D}$ is separable (Mokobodzki, cf. [4] exercise 5.9).

Let us emphasize nevertheless that a D-gradient does not give by image a D-gradient for the image structure :

Let $S_X = (\mathbb{R}^d,\ \mathcal{B}(\mathbb{R}^d),\ X_*m,\ \mathbb{D}_X,\ \Gamma_X)$ be the image of S by $X \in \mathbb{D}^d$, then the formula $\mathbb{E}[D[F \circ X] \mid X = x]$ does not define a D-gradient for X_*S because

$$(*) \begin{cases} < \mathbb{E}[D[F \circ X] \mid X = x]\ ,\ \mathbb{E}[D[F \circ X] \mid X = x] >_H \\ \neq \ \ \mathbb{E}[< D[F \circ X], D[F \circ X] >_H \ \mid X = x] \qquad (= \Gamma_X[F](x)). \end{cases}$$

A D-gradient is not a canonical notion, there is latitude in its definition. The space H in particular may be chosen in different ways depending on what is the most convenient and simple in the examples.

The Feyel-la-Pradelle derivative is a particular case of D-gradient which is canonical in the case the measure m is Gaussian and W a vector space (cf [5]). It can be generalized to non-Gaussian cases by taking for H the space $L^2(\hat{W}, \hat{\mathcal{W}}, \hat{m})$ where $(\hat{W}, \hat{\mathcal{W}}, \hat{m})$ is a copy of $(W, \mathcal{W}, m)$ (cf [5] and [4] chapter III§2 and chapter VII§1). With respect to our question, it has the same weakness as any D-gradient of being not preserved by images because of the inequality (*).

2 The measure valued gradient

The object we shall define possesses, like the D-gradient, some latitude in its definition and depends on an auxiliary Hilbert space.

We suppose $S = (W,\ \mathcal{W},\ m,\ \mathbb{D},\ \Gamma)$ admits a D-gradient defined with the separable Hilbert space H. For our purpose, let us recall the notion of *Gaussian white noise measure* based on a measurable space $(E, \mathcal{F})$ with associated positive measure μ.

Definition 1. *Let μ be a bounded positive measure on the measurable space $(E, \mathcal{F})$. A Gaussian white noise measure ν on $(E, \mathcal{F})$ defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with associated positive measure μ is a map from $\mathcal{F}$ into $L^2(\Omega, \mathcal{A}, \mathbb{P})$ s. t.*

(j) $\forall A \in \mathcal{F},\ \nu(A)$ *is a centered Gaussian r.v. with variance $\mu(A)$,*

(jj) $A \mapsto \nu(A)$ *is σ-additive in $L^2(\Omega, \mathcal{A}, \mathbb{P})$,*

(jjj) *if $A_1, \ldots, A_k \in \mathcal{F}$ are pairwise disjoint the r.v. $\nu(A_1), \ldots, \nu(A_k)$ are independent.*

Then ν extends uniquely from $\mathcal{F}$ to $L^2(E, \mathcal{F}, \mu)$ and we write $\nu(f)$ for $f \in L^2(E, \mathcal{F}, \mu)$. Given a measured space $(E, \mathcal{F}, \mu)$ such that $L^2(E, \mathcal{F}, \mu)$ is separable such a white noise measure may be constructed as a classical Wiener integral in the following way : let us take $(\Omega, \mathcal{A}, \mathbb{P}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathcal{N}(0,1))^{\mathbb{N}}$ so that the coordinate mappings (g_n) are i.i.d. Gaussian reduced r.v. Then for $f \in L^2(E, \mathcal{F}, \mu)$, we can put

$$\nu(f) = \sum_n (f, \xi_n)_{L^2(\mu)}\, g_n$$

where (ξ_n) is an orthonormal basis of $L^2((E, \mathcal{F}, \mu)$. If $L^2(E, \mathcal{F}, \mu)$ is no more supposed to be separable, such a white noise measure may be constructed as Gaussian process indexed by $\mathcal{F}$ by Kolmogorov theorem.

The positive measure μ associated with the white noise measure ν will be often denoted by the symbolic notation $\mathbb{E}_{\mathbb{P}}[(d\nu)^2]$.

Similarly, given on $(E, \mathcal{F})$ a symmetric matrix of bounded measures $\begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{12} & \mu_{22} \end{pmatrix}$ such that $\begin{pmatrix} \mu_{11}(A) & \mu_{12}(A) \\ \mu_{12}(A) & \mu_{22}(A) \end{pmatrix}$ be positive $\forall A \in \mathcal{F}$, we can define a bivariate white noise measure which to each $A \in \mathcal{F}$ associates a pair of Gaussian variables $(\nu_1(A), \nu_2(A))$ satisfying properties analogous to (j), (jj), (jjj). Such a bivariate white noise may be transformed in different ways :

a) It may be multiplied by a function $\varphi \in L^2(E, \mathcal{F}, \mu_{11} + \mu_{22})$:

$$(\varphi\nu)(A) = \begin{pmatrix} \int_A \varphi \, d\nu_1 \\ \int_A \varphi \, d\nu_2 \end{pmatrix}$$

is a bivariate white noise measure with associated matrix $\varphi^2 \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{12} & \mu_{22} \end{pmatrix}$.

b) For $x = (x_1, x_2) \in \mathbb{R}^2$ we can define the scalar white noise measure $(x, \nu) = x_1\nu_1 + x_2\nu_2$ whose associated measure is $x^t \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{12} & \mu_{22} \end{pmatrix} x$.

c) As a mixing of a) and b) let $\psi = (\psi_1, \psi_2)$ be in $L^2(E, \mathcal{F}, \mu_{11} + \mu_{22}; \mathbb{R}^2)$. We can define the scalar white noise measure (ψ, ν) by

$$(\psi, \nu)(A) = \int_A \psi_1 d\nu_1 + \int_A \psi_2 d\nu_2$$

whose associated positive measure is

$$\psi^t \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{12} & \mu_{22} \end{pmatrix} \psi = \psi_1^2\mu_{11} + 2\psi_1\psi_2\mu_{12} + \psi_2^2\mu_{22}.$$

More generally, we will need a notion of white noise measure with Hilbertian values :

Definition 2. *Given a bounded positive measure μ on a measurable space $(E, \mathcal{F})$ and a separable Hilbert space H, we call H-*valued white noise measure *defined thanks to the auxiliary probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with positive measure μ, a map from $\mathcal{F}$ into $L^2((\Omega, \mathcal{A}, \mathbb{P}); H)$ such that :*

(α) $\forall A \in \mathcal{F}$, $\forall h \in H$, $< \nu(A), h >_H$ *is a centered Gaussian variable with variance $\mu(A)\|h\|_H^2$*

(β) $A \in \mathcal{F} \longrightarrow \nu(A)$ *is σ-additive in $L^2((\Omega, \mathcal{A}, \mathbb{P}); H)$*

(γ) If $A_1, \ldots, A_k \in \mathcal{F}$ are pairwise disjoint, $\forall h \in H$ the r.v. $< \nu(A_1), h >, \ldots, < \nu(A_k), h >$ are independent.

Such ν naturally extends to functions $f \in L^2(E, \mathcal{F}, \mu)$ and $\forall h \in H$ $< \nu(f), h >_H$ is a centered Gaussian variable with variance $\int f^2 d\mu \|h\|_H^2$.

To construct such a ν, let us consider a sequence of independent copies ν_n of a real white noise measure on $(E, \mathcal{F})$ with associated positive measure μ, and for $f \in L^2(E, \mathcal{F}, \mu)$ let us put

$$\nu(f) = \sum_n \nu_n(f)\chi_n$$

where χ_n is a complete orthonormal system of H.

Similarly to the bivariate case, such a ν may be transformed in different ways.

a) multiplying by $\varphi \in L^2(E, \mathcal{F}, \mu)$

$$(\varphi\nu)(A) = \int 1_A \, \varphi \, d\nu$$

$\varphi\nu$ is a H-valued white noise measure, s.t. $\forall f \in L^\infty(E, \mathcal{F}, \mu)$, $\forall h \in H$

$$\mathrm{var}[< (\varphi\nu)(f), h >_H] = \int f^2 \varphi^2 \, d\mu . \|h\|_H^2.$$

b) For $x \in H$, we can define the scalar white noise measure (x, ν) by

$$(x, \nu)(A) = < x, \nu(A) >_H \,.$$

whose associated positive measure is $\|x\|^2 \mu$.

c) For $\psi \in L^2((E, \mathcal{F}, \mu); H)$ we can define the scalar white noise measure (ψ, ν) with associated positive measure $\|\psi\|_H^2 . \mu$ in the following way :

if ψ is decomposable $\psi = \sum_{i=1}^k \psi_i(w).h_i$ then we put

$$(\psi, \nu) = \sum_{i=1}^{k} \psi_i(w).(h_i, \nu)$$

where (h_i, ν) is defined in b). The associated positive measure is

$$\sum_{ij} \psi_i(w)\psi_j(w) < h_i, h_j >_H .\mu = \| \sum_{i=1}^{k} \psi_i(w) h_i \|_H^2 . \mu.$$

For the general case, let ψ_n be decomposable s.t. $\psi_n \to \psi$ in $L^2((E, \mathcal{F}, \mu); H)$, then we put

$$(\psi, \nu)(A) = \lim_n (\psi_n, \nu)(A) \text{ in } L^2(\Omega, \mathcal{A}, \mathbb{P}).$$

After these preliminaries, we can propose an answer to our initial question: let us consider an error structure $S = (W,\ \mathcal{W},\ m,\ \mathbb{D},\ \Gamma)$ admitting a D-gradient D constructed with the help of the separable Hilbert space H.

To any $X \in \mathbb{D}$ we shall associate a real white noise measure that will be called its **measure-valued gradient** with satisfactory properties by image.

Definition 3. *Let ν be an H-valued white noise measure on $(W, \mathcal{F})$ with associated positive measure m. Let $X \in \mathbb{D}$, and let DX be its D-gradient constructed with the Hilbert space H. The scalar white noise measure*

$$(DX, \nu)$$

defined as in c) above, will be called the measure-valued gradient of X and denoted

$$d_G X.$$

Thus $\forall f \in L^2(W, \mathcal{W}, m)$ we have

$$d_G X(f) \; \left(= \int f d_G X\right) \; =< DX, \nu(f) >_H .$$

Similarily if $X \in \mathbb{D}^d$ its measure-valued gradient is defined as the column-vector of the measure-valued gradients of its components. It is therefore an $\mathbb{R}^d$-valued white noise measure[1].

Proposition 1. *Let $X \in \mathbb{D}$. Let us denote*

$$\mathbb{E}_{\mathbb{P}}(d_G X)^2$$

the associated positive measure of $d_G X$. We have $\mathbb{E}_{\mathbb{P}}(d_G X)^2 << m$ and

$$\frac{\mathbb{E}_{\mathbb{P}}(d_G X)^2}{dm} = \Gamma[X].$$

Proof. Let $f \in L^\infty(W, \mathcal{W}, m)$. The Gaussian r.v. $\int f d_G X =< DX, \nu(f) >_H$ is defined on $(\Omega, \mathcal{A}, \mathbb{P})$ and has as variance $\int f^2 < DX, DX >_H dm$ by the construction c) defining (DX, ν). Hence, $\mathbb{E}_{\mathbb{P}}(d_G X)^2$ has a density with respect to m equal to $\Gamma[X]$. Q.E.D.

Similarly, if $X \in \mathbb{D}^d$

$$\mathbb{E}_{\mathbb{P}}(d_G X (d_G X)^t) = \underline{\underline{\Gamma}}[X, X^t].m.$$

where $\underline{\underline{\Gamma}}[X, X^t]$ is the matrix with elements $\Gamma[X_i, X_j]$.

Proposition 2.
a) $\forall X \in \mathbb{D}, \quad \forall F \in Lip \qquad d_G(F \circ X) = F'(X) d_G X$
b) $\forall X \in \mathbb{D}^d, \quad \forall F \in \mathcal{C}^1 \cap Lip(\mathbb{R}^d) \qquad d_G(F \circ X) = \sum_{i=1}^d F_i'(X) \, d_G X_i.$

Proof. These properties are straightforward from the corresponding ones of the D-gradient.

[1] The G of $d_G X$ is for Gauss who may be considered as the founder of error propagation calculus cf [1].

3 Images

Let us look now at what happens by image. Let $X \in \mathbb{D}^d$ and let $S_X = (\mathbb{R}^d,\ \mathcal{B}(\mathbb{R}^d),\ X_*m,\ \mathbb{D}_X,\ \Gamma_X)$ be the image by X of the error structure $S = (W,\ \mathcal{W},\ m,\ \mathbb{D},\ \Gamma)$.

For $F \in \mathbb{D}_X$, to define $d_G F$ we put

$$d_G F = X_*(d_G(F \circ X))$$

i.e. $d_G F$ is the image by X of the white noise measure $d_G(F \circ X) = (D(F \circ X), \nu)$. It is defined as a usual image of measure by

$$(d_G F)(A) = (d_G(F \circ X))(X^{-1}(A)) \quad \forall A \in \mathcal{B}(\mathbb{R}^d)$$

or $\int u^t.d_G F = \int u^t \circ X.d_G F \circ X$ for $u \in L^\infty(X_*m, \mathbb{R}^d)$.

Similarly if $\Phi = (\Phi_1, \ldots, \Phi_k) \in \mathbb{D}^k$, $d_G\Phi$ is defined as the column vector $(d_G\Phi_i)$.

Proposition 3.

a) $\forall F \in \mathbb{D}_X$ the positive measure associated to $d_G F$ is absolutely continuous w.r. to X_*m and

$$\frac{\mathbb{E}_{\mathbb{P}}(d_G F)^2}{dX_*m} = \Gamma_X[F].$$

b) $\forall F \in \mathcal{C}^1 \cap Lip(\mathbb{R}^d)$

$$d_G F = \nabla F^t.d_G I$$

where ∇F is the usual gradient of F and I is the identity map from $\mathbb{R}^d$ onto $\mathbb{R}^d$which belongs to $(\mathbb{D}_X)^d$. The $\mathbb{R}^d$-valued white noise measure $d_G I = X_* d_G X$ has for associated positive matrix of measures $(\Gamma_X[I_i, I_j].X_*m)_{ij}$.

Proof. a) From the fact that $\mathbb{E}_{\mathbb{P}}[(d_G(F \circ X))^2] = \Gamma[F \circ X].m$, the image of the white noise measure $d_G(F \circ X)$ by X has for associated positive measure $\mathbb{E}[\Gamma[F \circ X]|X = x].X_*m$ because of the definition of the conditional expectation, i.e. $\Gamma_X[F].X_*m$. This part of the proposition shows that the property of proposition 1 is preserved by image.

b) We know by proposition 2 that if $F \in \mathcal{C}^1 \cap Lip$

$$d_G(F \circ X) = (\nabla F)^t \circ X\ d_G X$$

the result follows taking the image. Q.E.D.

Let us denote $L^2(\mathbb{R}^d, \underline{\underline{\Gamma}}_X[I].X_*m)$ the space of d-uples of functions $v = (v_1, \ldots, v_d)$ defined on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ equipped with the norm given by $\|v\|^2 = \int v^t \underline{\underline{\Gamma}}_X[I]v\ dX_*m$. We obtain the main result of our study :

Proposition 4. *For every $F \in \mathbb{D}_X$, there exists an element of $L^2(\mathbb{R}^d, \underline{\underline{\Gamma}}_X[I].X_*m)$ denoted $\nabla_X F$ such that*

$$d_G F = (\nabla_X F)^t d_G I.$$

We have $\underline{\underline{\Gamma}}_X[F] = (\nabla_X F)^t \underline{\underline{\Gamma}}_X[I] \nabla_X F$ and on the initial structure we have also

$$d_G(F \circ X) = (\nabla_X F)^t \circ X \; d_G X.$$

Proof. Let (F_n) be a sequence of $\mathcal{C}^1 \cap Lip$ functions converging to F in $\mathbb{D}_X$. Denoting $\mathcal{E}_X$ the Dirichlet form of the structure S_X, we have

$$\mathcal{E}_X[F_n - F_m] = \frac{1}{2} \int \nabla(F_n - F_m)^t \underline{\underline{\Gamma}}_X[I] \nabla(F_n - F_m) \, dX_*m$$

hence ∇F_n converges in $L^2(\mathbb{R}^d, \underline{\underline{\Gamma}}_X[I].X_*m)$. Its limit ξ doesn't depend on the used sequence. For all $u \in L^\infty(\mathbb{R}^d, X_*m)$ the Gaussian variables $\int u \; d_G F_n = \int u (\nabla F_n)^t \; d_G I$ converge in $L^2(\Omega, \mathcal{A}, \mathbb{P})$ to $\int u \, \xi^t \, d_G I$ in other words the Gaussian variables $\int u \circ X \; d_G F_n \circ X$ converge in $L^2(\Omega, \mathcal{A}, \mathbb{P})$ to $\int u \circ X \, \xi^t \circ X \; d_G X$. It follows that $d_G F = \xi^t \, d_G I$ and $d_G(F \circ X) = \xi^t \circ X \; d_G X$. Q.E.D.

For a function $U \in (\mathbb{D}_X)^p$ with values in $\mathbb{R}^p$ we denote $\nabla_X U$ the matrix of the ∇_X of its components. We obtain a differential calculus :

Proposition 5. *Let U be a map from $\mathbb{R}^d$ into $\mathbb{R}^p$ such that $U \in (\mathbb{D}_X)^p$ and V a map from $\mathbb{R}^p$ into $\mathbb{R}^q$ such that $V \circ U \in (\mathbb{D}_X)^q$ and $V \in (\mathbb{D}_{U \circ X})^q$. Then*

$$(\nabla_X (V \circ U))^t = (\nabla_{U \circ X} V)^t \circ U . (\nabla_X U)^t.$$

Proof. By proposition 4 applied to U we have

$$d_G U = (\nabla_X U)^t \, d_G I_d, \qquad d_G(U \circ X) = (\nabla_X U)^t \circ X \, d_G X,$$

by proposition 4 applied to $V \circ U$ we have

$$d_G(V \circ U) = (\nabla_X (V \circ U))^t \, d_G I_d, \qquad d_G(V \circ U \circ X) = (\nabla_X (V \circ U))^t \circ X \, d_G X,$$

now by proposition 4 applied to V on the image structure by $U \circ X$ we have

$$d_G V = (\nabla_{U \circ X})^t \, d_G I_p, \qquad d_G(V \circ U \circ X) = (\nabla_{U \circ X} V)^t \circ U \circ X \, d_G(U \circ X).$$

It follows that

$$(\nabla_X(V\circ U))^t\circ X=(\nabla_{U\circ X}V)^t\circ U\circ X.(\nabla_X U)^t\circ X$$

equality in the space $L^2(E,\mathcal{F},\underline{\underline{\Gamma}}[X].m)$ and

$$(\nabla_X(V\circ U))^t=(\nabla_{U\circ X}V)^t\circ U.(\nabla_X U)^t$$

equality in the space $L^2(\mathbb{R}^d,\underline{\underline{\Gamma}}_X[I].X_*m)$. The argument comes therefore from the fact that the notions are defined thanks to images of measures. Q.E.D.

Let M_X be a measurable square root (non necessarily positive) of the matrix $\underline{\underline{\Gamma}}_X[I]$, i.e. such that $M_X^tM_X=\underline{\underline{\Gamma}}_X[I]$.

Corollary *For $F\in\mathbb{D}_X$ let us define*

$$D_XF=(\nabla_XF)^tM_X^t$$

then D_X is a Dirichlet-gradient for the image structure S_X defined with the Hilbert space $\mathbb{R}^d$.

Proof. $(D_XF,D_XF)_{\mathbb{R}^d}=(\nabla_XF)^t\underline{\underline{\Gamma}}_X[I]\nabla_XF$ which is equal to $\underline{\underline{\Gamma}}_X[F]$ by proposition 4. Hence D_X is a D-gradient for S_X. Q.E.D.

4 Example

Let us consider the classical Wiener space $(W,\mathcal{W},m)$ with $W=\mathcal{C}_0[0,1]$, $\mathcal{W}$ its Borel σ-field and m the Wiener measure equipped by the Ornstein-Uhlenbeck structure $(W,\mathcal{W},m,\mathbb{D},\Gamma)$ characterized by

$$\forall f\in L^2[0,1],\qquad \Gamma[\int f\,dw]=\|f\|^2_{L^2}$$

the space $\mathbb{D}$ is usually denoted $D_{2,1}$ or $\mathbb{D}^2_1$ (cf [4], [7], [8], [9]). We consider the Feyel-la-Pradelle gradient denoted $\#$, it is a linear map from $\mathbb{D}$ into $L^2(m,L^2(\hat{W},\hat{\mathcal{W}},\hat{m}))$ where $(\hat{W},\hat{\mathcal{W}},\hat{m})$ is a copy of $(W,\mathcal{W},m)$. Thanks to the functional calculus it is characterized by its values on the first chaos :

$$\forall f\in L^2[0,1],\qquad (\int f\,dw)^\#=\int f\,d\hat{w}$$

the Hilbert space H is $L^2(\hat{W},\hat{\mathcal{W}},\hat{m})$. Let (Z_n) be an orthonormal basis of $L^2(W,\mathcal{W},m)$ for instance composed with a basis of each Wiener chaos,

$(\hat{Z}_k)$ the corresponding basis of $L^2(\hat{W}, \hat{\mathcal{W}}, \hat{m})$ and let $g_{n,k}$ be i.i.d. reduced Gaussian variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

Putting for $Y \in L^2(W, \mathcal{W}, m)$

$$\int Y \, d\nu = \sum_{k,n} \mathbb{E}_m[Y Z_n] \hat{Z}_k \, g_{n,k}$$

defines according to definition 2 an H-valued white noise measure on $(W, \mathcal{W})$ with positive measure m.

If $X \in \mathbb{D}$ according to definition 3 for $Y \in L^\infty(W, \mathcal{W}, m)$

$$\int Y d_G X = \sum_{k,n} \mathbb{E}_m[Y Z_n \mathbb{E}_{\hat{m}}[X^{\#} \hat{Z}_k]] g_{n,k}$$

and we have

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[(\textstyle\int Y \, d_G X)^2] &= (\textstyle\sum_{k,n} \mathbb{E}_m[Y Z_n \mathbb{E}_{\hat{m}}[X^{\#} \hat{Z}_k]])^2 \\ &= \textstyle\sum_k \mathbb{E}_m[(Y \mathbb{E}_{\hat{m}}[X^{\#} \hat{Z}_k])^2] = \mathbb{E}_m[Y^2 \Gamma[X]]\end{aligned}$$

so that the positive measure associated with $d_G X$ is indeed $\Gamma[X].m$ and the study applies.

These results mean that a differential calculus may be defined on an error structure and its images, satisfying the expected coherence property, which coincides with the usual differential calculus on $\mathcal{C}^1 \cap Lip$ functions but exists also by completion for any function in the Dirichlet spaces of the images structures, coherence being preserved, thanks to the fact that the image of a gradient is now defined as the usual image of a measure. The tools introduced here are not intrinsic, this would be an interesting program to geometrize them. But in the applications, for studying the sensitivity of stochastic models, we are mostly concerned with computations in situations where an error structure is defined on the Wiener space (e.g. the Ornstein-Uhlenbeck structure or a generalized Mehler-type structure) or on the Poisson space or both, and all is about images of this structure (cf. [3]) the preceding study is relevant from this point of view.

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